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INTEGRATION OF RICCATI'S EQUATION.

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For illustrating important points of analysis, this Equation is adapted to a place in the Calculus, similar to that of the "problem of the lights" in Algebra. Its "integrable cases" were first pointed out by Count Riccati in the Leipsic Acts of 1732.

RICCATI'S EQUATION, $$\frac{du}{dx} + bu^2 = cx^m. \tag{1}$$

FIRST INTEGRABLE CASE.

When $m = 0$, the last term becomes c; and $\frac{du}{c - bu^2} = dx$. Integrating this, and then reducing, with C as the arbitrary constant, and ε as the base of hyperbolic logarithms, we obtain the integral (2) when b and c have like signs, and the integral (3) when their signs are unlike.

$$u = \left(\frac{c}{b}\right)^{\frac{1}{2}} \frac{C\varepsilon^{2x(bc)^{\frac{1}{2}}} - 1}{C\varepsilon^{2x(bc)^{\frac{1}{2}}} + 1}. \tag{2}$$

$$u = \left(-\frac{c}{b}\right)^{\frac{1}{2}} \tan\left(C + x(-bc)^{\frac{1}{2}}\right). \tag{3}$$

SECOND INTEGRABLE CASE.

When $m = -2$, let the value $u = \frac{y}{x} + \frac{1}{2bx}$ be substituted in (1), which leads to the result (4), of which the integral y is evidently found from (2) or (3) by changing x into $\log x$, and c into $c + \frac{1}{4b}$.

$$\frac{xdy}{dx} + by^2 = \left(c + \frac{1}{4b}\right), \quad u = \frac{y}{x} + \frac{1}{2bx}. \tag{4}$$

OTHER INTEGRABLE CASES.

By the reciprocal and direct transformations below, it was long since proved that Riccati's Equation is always integrable in a *continued fraction*, or in finite terms, when its exponent m or m' is a term in either of the series (5) or (6):—

$$m = \frac{-4i}{2i-1} = 0, -4, -\frac{8}{3}, -\frac{12}{5}, -\frac{16}{7}, -\frac{20}{9}, \ldots, -2; \quad (5)$$

$$m' = \frac{-4i}{2i+1} = 0, -\frac{4}{3}, -\frac{8}{5}, -\frac{12}{7}, -\frac{16}{9}, -\frac{20}{11}, \ldots, -2. \quad (6)$$

$$\frac{m}{2m+4} \text{ or } \frac{-m'}{2m'+4} = i = 0, 1, 2, 3, 4, 5, \ldots, \infty;$$

$$m+2 = n = \frac{-2}{2i-1}; \quad m'+2 = n' = \frac{2}{2i+1}.$$

Among the *singular features*, the negative sum equals the product, or $-(m+m') = mm' = \frac{16i^2}{4i^2-1}$; also $m' = \frac{-m}{1+m}$ and $m = \frac{-m'}{1+m'}$, for the same value of i. A change of the sign of i likewise changes the value of m from either one of the series (5), (6) to the other.

RECIPROCAL TRANSFORMATION.

In equation (1) let us substitute $u = \frac{1}{u_1}$, and $x = x_1^{\frac{1}{m+1}}$; the result will still have Riccati's form:

$$\frac{du_1}{dx_1} + \frac{c}{m+1} u_1^2 = \frac{b}{m+1} x_1^{\frac{-m}{m+1}}. \quad (7)$$

By this operation, the given exponent m is changed to $\frac{-m}{m+1}$, that is, from one of the series (5), (6) to the other, as from $-\frac{12}{7}$ to $-\frac{12}{5}$. If u in (1) be large, its reciprocal in (7) will evidently be small, and conversely; since $uu_1 = 1$. To repeat the transformation (7) would merely return to the equation (1). It is remarkable that Riccati's form alike re-appears for the reciprocal of u in (7), and for the reciprocal of x in the following equation (8). For which, let us substitute in (1) $xz = 1$, or $x = \frac{1}{z}$, and $u = \frac{z}{b} + z^2 y$. The general result so found for the reciprocal of x, is also integrable at once when $m = -4$.

$$\frac{dy}{dz} - by^2 = -cz^{-m-4}. \quad (8)$$

DIRECT TRANSFORMATION.

By the usual substitution of $u = \frac{1}{bx} + \frac{1}{x^2 u_1}$, in equation (1), we find first

$$\frac{x^2 du_1}{dx} + cx^{m+4} u_1^2 = b.$$

This result is brought into Riccati's form by making $x = x_1^{\frac{1}{m+3}}$:

$$\frac{du_1}{dx_1} + b_1 u_1^2 = c_1 x_1^{m_1}. \tag{9}$$

Here

$$b_1 = \frac{c}{m+3}, \quad c_1 = \frac{b}{m+3}, \quad m_1 = -\frac{m+4}{m+3},$$

$$b_2 = \frac{c_1}{m_1+3}, \quad c_2 = \frac{b_1}{m_1+3}, \quad m_2 = -\frac{m_1+4}{m_1+3}, \text{ etc.}$$

The process can thus be continued to any extent by alteration and substitution. When m belongs in the series (5),

$$m = \frac{-4i}{2i-1}, \quad m_1 = -\frac{m+4}{m+3} = \frac{-4(i-1)}{2(i-1)-1}.$$

That is, the direct transformation from (1) to (9) has diminished i by 1. Hence when i is a positive integer, the repetition of the process i times, would reduce the exponent to 0, which renders the equation integrable in the form of (2) or (3).

On the contrary when m belongs in the series (6), we find

$$m' = \frac{-4i}{2i+1}, \quad m'_1 = -\frac{m'+4}{m'+3} = \frac{-4(i+1)}{2(i+1)+1}. \tag{10}$$

Here each direct transformation increases i by 1, or removes one step from left to right, as from $-\frac{8}{5}$ to $-\frac{12}{7}$. Hence if the given exponent m occurs in (6), the reciprocal transformation will first change it to (5) where direct transformations will reduce the exponent to 0. By these or similar operations, the integrable cases were first resolved by continued fractions, which have opened the way to the complete solution.

INTEGRATION.

Comparing equation (1) with the linear equation $\frac{du}{dx} + Pu = Q$, let us equate the second terms, making $bu^2 = Pu$, or $bu = P$. The well known formula for

solving the linear equation contains an exponential, here denoted by w, such that $\int P dx = \log w$, or $P dx = \frac{dw}{w}$. Therefore equating the two expressions for P, and substituting the result in equation (1) we find, by making $m + 2 = n$,

$$u = \frac{dw}{bw dx}, \quad x^2 \frac{d^2w}{dx^2} = bcx^n w. \tag{11}$$

To discover the form of the integral, we adopt the usual method of first placing $\frac{d^2w_0}{dx^2} = 0$, whence by integration $w_0 = Px + Q$.

To satisfy equation (11), these constants P, Q may next be regarded as variables or variable series, assumed as follows: If $w_1 = Px$, and $w_2 = Q$,

$$w_1 = C_1 x [1 + A\ (bcx^n) + B\ (bcx^n)^2 + \ldots],$$
$$w_2 = C_2\ [1 + A'(bcx^n) + B'(bcx^n)^2 + \ldots].$$

The two series represent two "particular solutions," as the theory requires. Each series, as w_1, is to be substituted separately in equation (11) to determine $A, B, \ldots$ by the common "method of indeterminate co-efficients."

$$w_1 = C_1 x \left\{ 1 + \frac{bcx^n}{n(n+1)} + \frac{(bcx^n)^2}{n(n+1)\,2n(2n+1)} + \frac{(bcx^n)^3}{n(n+1)\,2n(2n+1)\,3n(3n+1)} + \ldots \right\},$$

$$w_2 = C_2 \left\{ 1 + \frac{bcx^n}{n(n-1)} + \frac{(bcx^n)^2}{n(n-1)\,2n(2n-1)} + \frac{(bcx^n)^3}{n(n-1)\,2n(2n-1)\,3n(3n-1)} + \ldots \right\},$$

$$w = w_1 + w_2, \quad u = \frac{\frac{dw_1}{dx} + \frac{dw_2}{dx}}{b(w_1 + w_2)}. \tag{12}$$

This integral in series is entirely regular and convergent, but it has its failing cases. For when $n - 1 = 0$, or $2n - 1 = 0$, etc., the corresponding terms become infinite, and the arbitrary constant divides out from n.

ANOTHER FORM OF INTEGRAL.

Let the independent variable x be changed to z in equation (11) by making

$$x = z^{\frac{2}{n}},$$

$$\frac{dw}{dx} = \frac{n}{2} z^{\frac{m}{n}} \frac{dw}{dz},$$

$$\frac{x^2 d^2 w}{dx^2} = \frac{n^2 z^2}{4}\frac{d^2 w}{dz^2} + \frac{mnz}{4}\frac{dw}{dz},$$

$$\frac{d^2 w}{dz^2} + \frac{2i}{z}\frac{dw}{dz} + q^2 w = 0. \qquad (13)$$

Here $\frac{m}{n} = 2i$, where i may be positive or negative, and $-\frac{4bc}{n^2} = q^2$. Let us first suppose b,c in equation (1) to have unlike signs, so that q^2 is essentially positive. As equations (3) and (12) suggest, let us assume

$$w = v \cos(qz + C) + v' \sin(qz + C),$$

$$\frac{dw}{dz} = \left\{\frac{dv}{dz} + v'q\right\}\cos(qz + C) + \left\{\frac{dv'}{dz} - vq\right\}\sin(qz + C).$$

Substituting for w in equation (13), the total co-efficient of $\cos(qz + C)$ becomes zero, as authorized by the two unknown quantities v and v'; and this reduces the co-efficient of $\sin(qz + C)$ to zero. Thus we find the two simultaneous equations

$$\frac{d^2 v'}{dz^2} - 2q\frac{dv}{dz} + \frac{2i}{z}\left\{\frac{dv'}{dz} - qv\right\} = 0, \quad \frac{d^2 v}{dz^2} + 2q\frac{dv'}{dz} + \frac{2i}{z}\left\{\frac{dv}{dz} + qv'\right\} = 0.$$

Let $v'' = v' + kv$. Multiplying the latter of the two equations by k and adding to the former, we find when $k = \sqrt{-1}$,

$$\frac{d^2 v''}{dz^2} + \left\{\frac{2i}{z} + 2qk\right\}\frac{dv''}{dz} + \frac{2iqk}{z} v'' = 0. \qquad (14)$$

The expression for w above contains one arbitrary constant C, so that only one particular solution of (14) will be required to give the other C' by "the method of indeterminate co-efficients," assuming

$$v'' = v' + kv = C' z^{-i}\left\{1 + \frac{B}{z} + \frac{C}{z^2} + \frac{D}{z^3} + \ldots\right\}.$$

Substituting in 14,

$$B = \frac{-i(i-1)}{1 \cdot 2qk}, \quad C = \frac{-B(i+1)(i-2)}{2 \cdot 2qk}, \quad D = \frac{-C(i+2)(i-3)}{3 \cdot 2qk}, \text{ etc.}$$

The real terms of the series evidently give the value of v', and the imaginary terms divided by k give the value of v. And thus,

$$w = C' z^{-i}\left\{\frac{i(i-1)}{1 \cdot 2qz} - \frac{(i+2)\ldots(i-3)}{1 \cdot 2 \cdot 3\,(2qz)^3} + \ldots\right\}\cos(qz + C) \qquad (15)$$

$$+ C' z^{-i}\left\{1 - \frac{(i+1)\ldots(i-2)}{1 \cdot 2\,(2qz)^2} - \frac{(i+3)\ldots(i-4)}{1 \cdot 2 \cdot 3 \cdot 4\,(2qz)^4} - \ldots\right\}\sin(qz + C).$$

Referring now to the first of equations (11) and to subsequent derivatives of x and w with respect to z, we next find equation (16); and the quotient of (16) divided by (15) determines u, thus: —

$$\frac{dw}{bdx} = C' \frac{qnz^i}{2b} \times \tag{16}$$

$$\left\{ \begin{array}{l} \left\{ 1 - \frac{(i+2)\ldots(i+1)}{1.2(2qz)^2} + \frac{(i+4)\ldots(i-3)}{1.2.3.4(2qz)^4} - \ldots \right\} \cos(qz + C) \\ \qquad - \left\{ \frac{i(i-1)}{1.2qz} - \frac{(i+3)\ldots(i-2)}{1.2.3(2qz)^3} + \ldots \right\} \sin(qz + C) \end{array} \right\};$$

$$\frac{du}{dx} + bu^2 = cx^m; \tag{1}$$

$$m + 2 = n, \quad \frac{m}{n} = 2i, \quad u = \frac{dw}{bdx} \div w. \tag{17}$$

Also $z = x^{\frac{n}{2}}$, and $q^2 = \frac{-4bc}{2n}$. And the signs of b, c in equations (1), (15), and (16) are supposed to be *unlike* in the integral (17).

In case b, c in (1) have *like signs* the requisite changes in (15) and (16) are easily made; let $q = q' \sqrt{-1}$, and let $C' \sqrt{-1}$ be written in place of the arc C; also make this $(q'z + C')\sqrt{-1} = t\sqrt{-1}$. By changing the former sine and cosine to their exponential values, we have

$$\sin(qx + C) = \frac{\varepsilon^{-t} - \varepsilon^{+t}}{2\sqrt{-1}}, \quad \cos(qx + C) = \frac{\varepsilon^{-t} + \varepsilon^{+t}}{2}.$$

Substituting the new equivalents, and then dividing (16) by (15), we find that the factor $\sqrt{-1}$ first renders all the terms of $2q'z$ positive, and then disappears, leaving, when b and c have like signs, and $q'z + C' = t$,

$$N = \left\{ 1 + \frac{(i+2)\ \ldots\ (i-1)}{1\ .\ 2(2q'z)^2} + \ldots \right\} (\varepsilon^{-t} + \varepsilon^{+t}),$$

$$- \left\{ \frac{i(i+1)}{2q'z} - \ldots \right\} (\varepsilon^{-t} - \varepsilon^{+t}),$$

$$\Delta = \left\{ 1 + \frac{(i+1)\ \ldots\ (i-2)}{1\ .\ 2(2q'z)^2} + \ldots \right\} (\varepsilon^{-t} + \varepsilon^{+t}),$$

$$+ \left\{ \frac{i(i-1)}{2q'z} + \ldots \right\} (\varepsilon^{-t} - \varepsilon^{+t}),$$

$$u = -\frac{nq'z^{2i}}{2b} \frac{N}{\Delta}. \tag{18}$$

Thus the integral of Riccati's Equation is one of comparative simplicity. For by adding 1 to each factor of i in the denominator, or by simply changing the sign of i in the factors, we have the series of the numerator. The series are

evidently finite when i is an integer; but for all other values of i, they are infinite, and ultimately divergent, like well known series for the Gamma function. As the convergent parts of the latter function have long been used continuously, so may a limited number of terms of the Riccati series be correctly used for computing the true value of u, when i falls between the integer values.

[NOTE.—This important inference is further confirmed by the fact that an interpolation formula could certainly be so employed, and that the finite part of the series for w in equation (15) has precisely the form of an interpolation formula, deduced from actual values of u. For example, when $i - 4 = 0$, the series for w in equation (15) reduces to four terms, and the series (16) to five terms, which will determine the value of u, not only when $i = 4$, but also when $i = 3.2$, 3.3, 3.7, 3.8, or any value intermediate between 3 and 4. By first augmenting i, as shown by equation (10), the number of terms may be augmented, and u computed to any degree of accuracy.]

A DISCUSSION OF THE EQUATION OF THE SECOND DEGREE IN TWO VARIABLES.

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[CONCLUDED FROM PAGE 83.]

ELLIPSES CLASSIFIED.

(α). $R's\Delta < 0$. Since Δ is supposed negative, s is here positive. Distinguishing by the subscripts 1 and 2 the transverse and conjugate values, we have

$$e_1^2 = \frac{2R'}{R'+s}, \quad e_2^2 = \frac{2R'}{R'-s},$$

$$p_1'' = \sqrt{\frac{-4\Delta}{R'(s-R')^2}}, \quad p_2'' = \sqrt{\frac{+4\Delta}{R'(s+R')^2}}.$$

Whence it is seen that e_1 and p_1'' are real, while e_2 and p_2'' are imaginary. Hence, since to the real directrices and foci there corresponds a real value of e, the curve is real. A_1 and A_2 are both real and finite, and (15) shows that the curve has no point at a distance from the centre greater than A_1 or less than A_2. The two values of the semi-parameter are both real. This variety of the ellipse may therefore be described as having *real, finite, unequal axes.*

(β). $R's\Delta > 0$. This implies that s is negative. Hence e_1 and p_2'' are imaginary while e_2 and p_1'' are real. Hence the real value of the ratio e corresponds to the imaginary directrices while the imaginary value of e corresponds to